Enhanced Constraint Exploration on Fuzzy Quadratic Programming Problems

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Abstract: In this paper, we proposed a computational procedure by using fuzzy approach to find the optimal solution of quadratic programming problem. We divided the calculation of the optimal solution in two stages. In the first stage we determine the unconstrained minimization of the objective function of the fuzzy quadratic programming problem and check its feasibility. In the second stage, we are exploring the boundary of the feasible region from initial point to another point until the optimal point is achieved. A numerical example included in the support of this paper for illustration. Numerical results show the capability of an alternative method in order to obtain the optimal solution of the problems.

Keywords: Feasible Set; Fuzzy Optimal Solution; Positive Definite; Triangular Fuzzy Number; Quadratic Programming

1. Introduction

The theory of quadratic programming problems (QPP) is concerned with problems of constrained minimization where the constraint functions are linear, and the objective is a positive definite quadratic function. Although it represents a natural transition from theory of linear programming to the theory of nonlinear programming problem, there are some important differences between their optimal solution. If the optimum solution of QPP exists, then it is either an interior point or a boundary point which is not necessarily an extreme point of the feasible region. Constraint exploration is an iterative method that explore the feasible region of QPP from an initial point to another point in the feasible region of QPP until the optimal point is found [1, 2].

Li et al. [3] proposed a two-phase augmented Lagrangian method, called QSDPNAL, for solving convex quadratic semidefinite programming (QSDP) problems with constraints consisting of a large number of linear equality and inequality constraints, a simple convex polyhedral set constraint, and a positive semidefinite cone constraint. Later on, in 2018, a derivation of the classical block of symmetric Gauss-Seidel method to solve a positive define QPP from the optimization perspective has been proposed [4].

For general quadratically constrained quadratic programming (QCQP), Huang and Sidiropoulos [5] proposed a new algorithm and claimed that the method is different from any existing method, general or specialized, for such problems. The main ideas behind this proposed algorithm are any QCQP can be optimally solved, irrespective of non-convexity, consensus adopting the alternating direction method of multipliers can be used to solve general QCQPs, in such a way that each update requires to solve a number of QCQPs. A class of QCQP called a parametric active-set algorithm for quadratic programming, parallel QPP method to solve the dynamic programming problem, and heuristic method to solve the Integer QPP were proposed [6-8]. Several applications of QPP and methods for solving QPP can be found in [9-15]. However, the parameters in most actual practical applications such as portfolios, game theory, finite impulse response (FIR), design and control, logistics, etc. are approximate, vague and imprecise or in other words the exact values of the parameters are not known exactly in advance. These uncertainties can be cost of the objective function, and coefficients, the right-hand value in the set of constraints. Sometimes, these parameters have to be provided by the experts. Thus, the application of fuzzy numbers in this context...
of QPP is also a way of describing these vagueness mathematically as described in [1].
Bellman and Zadeh [16, 17] proposed the concept of decision making in fuzzy environment while Tanaka et.al [18] adopted this concept for solving mathematical programming problems. The first formulation of linear programming in fuzzy environment is proposed by [19]. Ammar and Khalifah [20] introduced the formulation of fuzzy portfolio optimization problem as a convex quadratic programming approach and gave an acceptable solution to such problem.

Despite the uncertainty can be presented in any part of the formulation, the focus of this work is to show it in all parameters involve in the QPP which can be dealt with by fuzzy numbers. An alternative method which is called the constraints exploration method is proposed to find the fuzzy optimal solution of QPP. By using the propose method the fuzzy optimal solution of QPP occurring in the real-life situations, can be easily obtained.

The rest of this paper is organized as follows. Section 2 introduces the basic concept of fuzzy numbers that be used in the context of QPP. Section 3 describes and characterize the formulation of the QPP and fuzzy QPP. Section 4 presents the proposed with fuzzy environment while Section 5 introduces the algorithm of the method. Numerical results are discussed in Section 6 and the conclusion of the work which ends this paper in Section 7.

2. Preliminaries
In this section, some necessary background and notations of fuzzy set theory are reviewed.

2.1 Basic Definition

Definition 1 [21]. The characteristic function $\mu_A$ of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in $X$. This function can be generalized to a function $\mu_A^+$ such that the value assigned to the element of the universal set $X$ falls within a specified range i.e. $\mu_A^+: X \to [0,1]$. The assigned value indicates the membership grade of the element in the set $A$. The function $\mu_A^+$ is called the membership function and the set $\bar{A} = \{(x, \mu_A^+(x)) : x \in X\}$ defined by $\mu_A^+$ for each $x \in X$ is called a fuzzy set.

Next three definitions regarding a triangular fuzzy number, corresponds $\alpha$-cuts, and non-negativity triangular fuzzy numbers are described [22].

Definition 2. A fuzzy number $\bar{A} = (a,b,c)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\bar{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & a \leq x \leq b \\ \frac{(x-c)}{(b-c)}, & b \leq x \leq c \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

and $\alpha$-cuts corresponding to $\bar{A} = (a,b,c)$ is given by [11]:

$$\bar{A}(\alpha) = [a_1(\alpha), a_2(\alpha)], \quad \alpha \in [0,1] \quad (2)$$

where $a_1(\alpha) = (b-a)\alpha + a$ and $a_2(\alpha) = -a - (c-b)\alpha + c$.

Definition 3. A triangular fuzzy numbers $(a,b,c)$ is said to be non-negative fuzzy number if and only if $a \geq 0$.

Definition 4. Let $\bar{A} = (a_1,b_1,c_1)$ and $\bar{B} = (a_2,b_2,c_2)$ be two triangular fuzzy numbers, then

a) $(\bar{A}) \leq (\bar{B})$ iff $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$.

b) $(\bar{A}) \pm (\bar{B})$ iff $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$.

2.2 Fuzzy Arithmetic

The following concepts and results are introduced from [21-23]. Let $\bar{A}(\alpha) = [a_{1\alpha}, a_{2\alpha}]$ and $\bar{B}(\alpha) = [b_{1\alpha}, b_{2\alpha}]$ be two closed, bounded, intervals of real numbers. If $*$ denotes addition, subtraction, multiplication, or division, then $[a_{1\alpha}, a_{2\alpha}] * [b_{1\alpha}, b_{2\alpha}] = \{ [\beta, \delta] \}$ where

$$[\beta, \delta] = \{a*b \mid a_{1\alpha} \leq a \leq a_{2\alpha}, b_{1\alpha} \leq b \leq b_{2\alpha}\} \quad (3)$$

If $*$ is division, we must assume that zero does not belong to $[b_{1\alpha}, b_{2\alpha}]$. We may simplify the above equation as follows:

1. Addition
$$[a_{1\alpha}, a_{2\alpha}] \oplus [b_{1\alpha}, b_{2\alpha}] = [a_{1\alpha} + b_{1\alpha}, a_{2\alpha} + b_{2\alpha}]$$
2. Subtraction
$$[a_{1\alpha}, a_{2\alpha}] \ominus [b_{1\alpha}, b_{2\alpha}] = [a_{1\alpha} - b_{1\alpha}, a_{2\alpha} - b_{2\alpha}]$$
3. Division
$$[a_{1\alpha}, a_{2\alpha}] \oslash [b_{1\alpha}, b_{2\alpha}] = \{ [\beta, \delta] \}$$

4. Multiplication
$$[a_{1\alpha}, a_{2\alpha}] \otimes [b_{1\alpha}, b_{2\alpha}] = [\beta, \delta]$$

where

$$\beta = \min\{a_{1\alpha} - b_{1\alpha}, a_{2\alpha} - b_{1\alpha}, a_{1\alpha} - b_{2\alpha}, a_{2\alpha} - b_{2\alpha}\} \quad (4)$$

and

$$\delta = \max\{a_{1\alpha} - b_{1\alpha}, a_{2\alpha} - b_{1\alpha}, a_{1\alpha} - b_{2\alpha}, a_{2\alpha} - b_{2\alpha}\} \quad (5)$$

The multiplication may be further simplified as follows. For $\bar{A} = (a,b,c)$ and $\bar{B} = (x,y,z)$ be a non-negative triangular fuzzy number, then

$$\bar{A} \otimes \bar{B} = \begin{cases} (ax,by,cz), & a \geq 0 \\ (ax,by,cz), & a < 0, c \geq 0 \\ (ax,by,cz), & c < 0 \end{cases} \quad (6)$$

Lemma [24]. Suppose that $f(x)$($x \in \mathbb{R}$) is an ordinary real valued function, and $\bar{A}$ be the set of all closed and bounded fuzzy numbers. If $\bar{f} = (n_1, n_2) \in \bar{A}$ then $\bar{f}$ satisfies:

1. $\{x \mid x \in \mathbb{R}, \bar{f}(x) = 1\} \neq \emptyset$
2. if we define \( f(\vec{r}) \oplus \bigcup_{\alpha(0)} f(\vec{r}_\alpha) \) then 
\[ (f(\vec{r}))_\alpha = f(\vec{r}_\alpha) = [\wedge_{x(\alpha)} f(x), \vee_{x(\alpha)} f(x)] \]
3. \( f(\vec{r}_\alpha) \in \vec{A} \)

3. Problem Formulation

A quadratic function on \( \mathbb{R}^n \) to be consider in this paper, which is defined by:
\[
f(x_1,\ldots,x_n) = \frac{1}{2} \sum_{i,j=1}^{n} d_{ij} x_i x_j + \sum_{j=1}^{n} c_j x_j + q
\]  
where \( q,c \) and \( d_{ij}, (i,j=1,\ldots,n) \) are constant scalar quantities can be written in vector-matrix notation as:
\[
f(x) = \frac{1}{2} x^T D x + c^T x + q
\]  
in which \( D = (d_{ij})_{n \times n}, c = (c_1,\ldots,c_n)^T \), and \( x = (x_1,\ldots,x_n)^T \).

Without loss of generality, we consider \( D \) to be a positive definite symmetric matrix and if \( D \) is a positive definite, then \( f(x) \) which is given by Equation (8) can be called as a positive definite quadratic function.

The set all feasible solution so-called the feasible region which will be considered in this paper, is a closed set defined by:
\[
F = \{ x \mid A x \leq b, x \geq 0 \}
\]  
where \( A \) is an \( (m \times n) \) matrix and \( b \) a vector in \( \mathbb{R}^m \).

Since \( f(x) \) given by Equation \( 7 \) is a positive definite quadratic function, then \( f(x) \) is strictly convex in \( x \), therefore \( f(x) \) attains a unique minimum at
\[
\vec{x}^*(\alpha) = -D^{-1}c
\]  
which is called un constrained minimum of \( f(x) \). As mentioned in Section 1, \( \vec{x}^*(\alpha) \) can be an interior point or boundary point of feasible region. However, there is one more possibility that is \( \vec{x}^*(\alpha) \) can be an exterior point. Therefore, if \( \vec{x}^*(\alpha) \in F \), then \( \vec{x}^*(\alpha) \) becomes the optimal solution of the QPP. Another advantages of strictly convexity properties of \( f(x) \) is that, if \( \vec{x}^*(\alpha) \) is an exterior point, then definitely, \( \vec{x}^*(\alpha) \) is the optimal solution of the considered problem is on the boundary of the feasible region. Therefore, \( \vec{x}^* \) must be located on one of the active or equality constraints or on the intersection of several active (equality) constraints [1, 2, 6].

In the conventional approach, the values of the parameter of QPP models must be well defined and precise. However, in real life this is not a realistic assumption. In the real problems there may exists uncertainty about the parameters. In such a situation the parameters of QPP with \( m \) fuzzy constraints and \( n \) fuzzy variables may be formulated as follows:

\[
\text{Minimize } Z(\vec{x}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}^+ x_i x_j + \sum_{j=1}^{n} c_j^+ x_j + q^-(\alpha)
\]  
subject to
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij})^- x_i x_j \leq \sum_{i=1}^{m} (b_i^-)_{\alpha}, (b_i^+)_{\alpha}
\]  
with all variables are non-negative, and \( \alpha \in [0,1] \).

4. Constraint Exploration Method

In this section, we describe how to search a point on the boundary of feasible region which becomes a candidate of the optimal solution to the problem in Equations (14)-(15).

Let us consider the fuzzy QPP with \( m = n = 2 \). Suppose that the fuzzy constraints are given by
The fuzzy QPP of the problem in Equations (16)-(19) is uniquely determined since there is one to one correspondence between the point and the respective \( \alpha \). Therefore, by substituting the above point into quadratic function in Equation (14), we can obtain the function with \( \alpha \) as the independent variable from which the unconstrained minimum of \( f(\alpha) \) can be achieved through minimizing \( f(\alpha) \) with respect to \( \alpha \). If \( \alpha^* \) denotes the unconstrained minimum of \( f(\alpha) \), then we obtain

\[
(\frac{C_1 + C_1 \alpha}{A_1}, \frac{C_2 \alpha^*}{B_1}),
\]

which refer to the fuzzy constrained minimum of \( f(x) \), subject to the equality constraint given by Equation (18). However, if the fuzzy constrained minimum in Equation (21) infeasible, then we search another fuzzy constrained minimum on the Equation (17). By similar procedure we get

\[
(\frac{C_2 + C_2 \alpha^*}{A_2}, \frac{C_2 \alpha^*}{B_2}).
\]

Now, if the point in Equation (22) also outside of feasible region, then absolutely the solution of considered problem will located in vertex or in the intersection of Equation (18) and Equation (19) and we evaluated as follows:

By substituting Equation (20) to Equation (19), i.e.,

\[
2x(\frac{C_1 + C_1 \alpha}{A_1}) + B_1(\frac{C_2 \alpha^*}{B_1}) = C_2
\]

we have

\[
\alpha^* = \frac{(C_2 A_1 + C_1 A_2)B_1}{(-A_2 B_1 + A_1 B_2)C_1}.
\]

Substituting Equation (24) to Equation (21), we get

\[
x^* = (\frac{C_1 B_1 - B_1 C_2}{A_1 B_1 + B_2 A_1} - \frac{C_2 A_1 + A_2 C_1}{-A_2 B_1 + B_2 A_1}),
\]

which refer to the fuzzy constrained minimum of \( f(x) \), subject to constraints in Equations (16) and (17).

5. The Outline of Algorithm

The results shown in previous section can be used to obtain an algorithm for finding the fuzzy optimal solution of QPP. The brief algorithm as follows:

1. Compute \( \bar{x}^{(0)} \), the unconstrained minimum of \( f(x) \) by using Equation (13).
2. If \( \bar{x}^{(0)} \) satisfies all the constraints provided by the problem, then stop, \( \bar{x}^{(0)} \) becomes the fuzzy optimal solution of the QPP. But if \( \bar{x}^{(0)} \not\in F \), then push all the indexes of the constraints violated by \( \bar{x}^{(0)} \) onto the set \( S \), where \( S = \{ j \mid A_j^T \bar{x} > b_j, j \in \{1, \ldots, m\} \} \) for further investigation.
3. Compute \( \bar{x}_j^* \), the fuzzy constrained minimum of \( f(x) \) subject to equality constraint \( j \) where \( j \in S \).

If \( \bar{x}_j^* \in F \) for one \( j \in S \), then \( \bar{x}_j^* \) is the fuzzy optimal solution of QPP and stop. Otherwise, search the fuzzy optimal solution of QPP which might be located on the equality or intersection of two and more equality violated constraints by \( \bar{x}^{(0)} \) according to the method explained in [1, 2].

6. Numerical Results

A few examples to show the capability of the fuzzy exploration method are shown below. All examples are taken from [26], where the objective function is to minimize the quadratic function and the constraint functions are consisting of two, one, three and six linear functions respectively. The first example given is as follows and all calculations are based on the algorithm in Section 5 are summarized in the Table 1.

Minimize \( z = (x_1 - 1)^2 + (x_2 - 2)^2 \) subject to

\[
-x_1 + x_2 = 1
\]

\[
x_1 + x_2 \leq 2
\]

and

\[
(x_1, x_2) \geq (0, 0).
\]

The fuzzy QPP of the problem in Equations (26)-(29) with \( \alpha \in [0,1] \) can be written as

Minimize \( \bar{Z}_\alpha(\bar{x}) = (\bar{x}_1 - 1)^2 + (\bar{x}_2 - 2)^2 \) subject to

\[
-\bar{x}_1 + \bar{x}_2 = 1
\]

\[
\bar{x}_1 + \bar{x}_2 \leq 2
\]

with all variables are non-negative.
Table 1 - Determination of fuzzy optimal solutions

<table>
<thead>
<tr>
<th>Example</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( n )</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>( m )</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>( x^{(0)}(\alpha) )</td>
<td>( \left[ \frac{\alpha + 1}{2}, -\alpha + 2 \right], \left[ \frac{3\alpha + 1}{2}, -\alpha + 3 \right] )</td>
</tr>
<tr>
<td></td>
<td>( S )</td>
<td>( {1,2} )</td>
</tr>
<tr>
<td></td>
<td>( x^*_i(\alpha), i \in S )</td>
<td>n/a</td>
</tr>
<tr>
<td></td>
<td>( x^*(\alpha) )</td>
<td>( 840 495,169 165 )</td>
</tr>
<tr>
<td></td>
<td>( \alpha )</td>
<td>( \left{ \frac{1}{2}, \frac{3}{2} \right} )</td>
</tr>
<tr>
<td>2</td>
<td>( n )</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>( m )</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>( x^{(0)}(\alpha) )</td>
<td>( \left[ \alpha, 2\alpha + 3 \right], \left[ \alpha, -2\alpha + 3 \right] )</td>
</tr>
<tr>
<td></td>
<td>( S )</td>
<td>( \mathbb{E} )</td>
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<tr>
<td></td>
<td>( x^*_i(\alpha), i \in S )</td>
<td>n/a</td>
</tr>
<tr>
<td></td>
<td>( x^*(\alpha) )</td>
<td>( \left[ \alpha, 2\alpha + 3 \right], \left[ \alpha, -2\alpha + 3 \right] )</td>
</tr>
<tr>
<td></td>
<td>( \alpha )</td>
<td>( (1,1) )</td>
</tr>
<tr>
<td>3</td>
<td>( n )</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>( m )</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>( x^{(0)}(\alpha) )</td>
<td>( \left[ \alpha + 4, -\alpha + 6 \right], \left[ \alpha + 2, -\alpha + 4 \right] )</td>
</tr>
<tr>
<td></td>
<td>( S )</td>
<td>( {3,5,6} )</td>
</tr>
<tr>
<td></td>
<td>( x^*_i(\alpha), i \in S )</td>
<td>( x_i(\alpha), x_5(\alpha) )</td>
</tr>
<tr>
<td></td>
<td>( x^*(\alpha) )</td>
<td>( x^* = x^*_5(\alpha) )</td>
</tr>
<tr>
<td></td>
<td>( \alpha )</td>
<td>( \left[ 840 495, 169 165 \right] )</td>
</tr>
</tbody>
</table>

7. Conclusions

In this paper we propose an iterative method called constraint exploration method that explore the feasible region from initial point to another feasible point until the optimal point is achieved. The optimum solution of QPP is either an interior point or a boundary point of the feasible region. The interior optimal solution or unconstrained optimal will be determined by differentiate of the objective function of the QPP with respect to all its variables and equate to zero. If the unconstrained optimal resides outside of the feasible region, the searching of the optimal point QPP problem will be focused on the region of violated constraints. Exploration the boundary of violated constraint from a point to another point until the optimal point is achieved. All the parameters of the QPP are considered in the form of fuzzy numbers. This method works in a fuzzy environment to handle the uncertainty of the parameters involved in the QPP problems. The fuzzy solution is characterized by fuzzy numbers, through the use of the concept of violation constraints by the fuzzy unconstrained optimal solution. By using this approach, the fuzzy optimal solution of quadratic programming problem which occurs in real life situation can be easily obtained. This method can be expanded further to solve the QPP involving many parameters, for instance, portfolio selection and finite impulse response. The numerical example presented is to show the capability of this method in obtaining the optimal solution of the fuzzy QPP.

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